Towards a Realistic Neutron Star Binary Inspiral

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An approach to general relativity based on conformal flatness and quasiequilibrium (CFQE) assumptions has played an important role in the study of the inspiral dynamics and in providing initial data for fully general relativistic numerical simulations of coalescing compact binaries. However, the regime of validity of the approach has never been established. To this end, we develop an analysis that determines the violation of the CFQE approximation in the evolution of the binary described by the full Einstein theory. With this analysis, we show that the CFQE assumption is significantly violated even at relatively large orbital separations in the case of corotational neutron star binaries. We also demonstrate that the innermost stable circular orbit (ISCO) determined in the CFQE approach for corotating neutron star binaries may have no astrophysical significance.

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a. Introduction The analysis of general relativistic binary neutron star processes is an important yet challenging endeavor. The importance of studying binary neutron star coalescence is rooted in observational astronomy; the process is considered to be a candidate for the sources of detectable gravitational waves as well as X- and γ-rays. Recently, there has been a large effort expended towards numerically simulating the inspiral and coalescence of compact objects in order to determine their signature gravitational waveforms in preparation for the detection of gravitational waves by the new generation of gravitational wave observatories, including LIGO, LISA, VIRGO, TAMA, and GEO. The complexity of the nonlinear Einstein field equations, the lack of exact symmetry of the system, and the coupling of different timescales in the process combine to make the study of the neutron star inspiral coalescence a major challenge.

A recent approach used to investigate the inspiral coalescence problem that has drawn much attention [1, 2, 3, 4, 5, 6, 7, 8, 9, 10] is based on the separation of two timescales, namely, the orbital motion timescale and the gravitational radiation timescale. This approach, which we refer to as the conformally flat quasiequilibrium (CFQE) approach, seeks to construct general relativistic configurations corresponding to two compact objects in circular orbit by assuming that the spacetime is essentially stationary with an approximate Killing vector (quasiequilibrium, QE). The slow secular evolution is driven by the gravitational radiation which has a much longer timescale than the orbital motion timescale, so that the spacetime is thought to be describable by a conformally flat (CF) spatial metric.

The CFQE approach leads to a two-phase description of the inspiral and coalescence process. In the "slowly inspiraling phase", the evolution is approximated by a time sequence (which we refer to as a CFQE-sequence) of CFQE configurations with the same rest mass and spin state (e.g. corotating or nonrotating). Each CFQE configuration in the time sequence is obtained by solving the four constraint equations plus one of the dynamical Einstein equations under the CF and QE assumptions (the QE assumption affects the constraint equations only indirectly). The inspiral is assumed to proceed along the sequence in a quasistationary manner on a timescale much longer than the orbital timescale. The slow secular evolution is driven by gravitational radiation, which can be calculated using e.g., the standard quadrupole formula (see, e.g., [11]). In the corotating case, when the orbit shrinks to a certain radius (which will depend on the equation of state (EOS) of the neutron stars), the CFQE configuration becomes secularly unstable, with the ADM mass of the system attaining a minimum among all configurations in the sequence. The orbit at this radius is referred to as the innermost stable circular orbit (ISCO). One expects the binary to become dynamically unstable at an orbital radius that is slightly less than this ISCO point. This ISCO point is therefore thought to be the point at which the system enters the "plunge phase", in which the subsequent merging occurs within an orbital timescale.

Besides providing a description of the inspiral coalescence process, the CFQE approach is important for another reason: it is expected to provide a starting point for numerical investigations in general relativity. The setting of initial data is a sensitive issue in the numerical integration of the Einstein equations. On the one hand, astrophysically relevant initial data must be used in order to make contact with gravitational wave observations. On the other hand, we are forced to use initial data in the late state of the inspiral, due to limited computational resources. As the CFQE approach gives configurations that satisfy the constraint equations of the Einstein theory, one is tempted to start numerical integration with a CFQE configuration near the ISCO point as determined in the CFQE approach, see, e.g., [12]. The danger in using the CFQE approach to determine the initial configuration for numerical integration is the unknown astrophysical relevance of the initial data, regard-

less of the fact that each CFQE configuration satisfies the constraints of general relativity and is therefore, in principle, a legitimate initial data set for numerical integrations. In order for the results of the numerical evolutions to be relevant to observations, e.g., the gravitational waves emitted in neutron star coalescences, we have to make sure that the initial data actually corresponds to a configuration in a realistic inspiral. The setting of physical initial data is a standard difficulty in numerical relativity and it is particularly acute in this case; if the initial data is picked to be a CFQE configuration in a regime where the CFQE-sequence is *not* expected to be valid, the initial data must be regarded as highly contrived and the resulting calculation will have no astrophysical significance.

In this paper we analyze the limitations of the CFQE approach, which suffers from two basic problems. The first problem is that the CF and QE assumptions violate the Einstein equations which, for this system, forbid both an exact Killing vector and a conformally flat metric at all times (the metric can only be chosen to be conformally flat at *one* point in time but not along the entire time sequence). The second problem is that the CFQE approach itself does not provide a way to estimate the error involved in its assumptions. A determination of its accuracy has to be carried out with a completely different analysis. Because of this, the regime of validity of the CFQE approach has never been determined. Here, we calculate the rate of increase of the accuracy of the CF and QE approximations with increasing initial neutron star separation. For the corotating binaries studied (with a $\Gamma = 2$ polytropic equation of state), we demonstrate that an initial separation of at least 6 neutron star radii is needed in order for the violation of the QE assumption to remain below 10% (in some reasonable measure defined below) after a short evolution (a fraction of an orbit). (The violation of the CF assumption is an order of magnitude smaller at this separation.) It is safe to conclude that, starting at a separation of 6 neutron star radii, the evolution (e.g., the orbital phase) of the system is completely different from that of the CFQE-sequence well before the CFQE ISCO separation (less than 3 neutron star radii) is reached.

In what follows, we present the results of our analysis of the CFQE-sequence approximation for binary, corotating neutron stars. The details of all calculations and the subsequent analysis of the numerical results, along with results on long timescale (e.g., multiple orbital period timescale) numerical evolutions will be reported elsewhere [13].

b. A General Relativistic Analysis of the CFQE Approximations for Corotating Neutron Star Binaries. To analyze the validity of the CFQE-sequence approximation, we compare it to solutions of the full Einstein equations whose initial Cauchy slice corresponds to specific CFQE configurations. We construct these corotating CFQE configurations following the algorithm detailed in [1]. The neutron stars we use are described by a polytropic EOS, $P = K\rho^\Gamma$, with $\Gamma = 2$ and $K = 0.0445 c^2/\rho_n$, where ρ_n is nuclear density (2.3 x 10^{14} g/cm^3). For this EOS, the maximum stable static neutron star configuration has an ADM mass of $1.79 M_\odot$ and a baryonic mass of $1.97 M_\odot$ ($M_\odot$ is 1 solar mass). In this paper, we use neutron stars whose baryonic mass M_0 is $1.49\, M_\odot$, which is approximately 75% that of the maximum stable configuration. The ADM mass of a single static neutron star for this configuration is $1.4 M_\odot$.

The corotational CFQE configurations with a fixed baryonic mass can be uniquely specified by the separation between the two neutron stars. We measure the spatial separation on a constant time hypersurface by the spatial geodesic distance $\ell_{1,2}$ between the points of maximum baryonic mass density in each of the neutron stars. This particular choice of measure of distance depends on the choice of time slicing. In order to compare with the CFQE-sequence approximation in an invariant manner, we use the same slicing condition (maximal slicing) as that used in CFQE-sequence approach. Four CFQE configurations denoted respectively NS-1 to NS-4 are used as initial data in our general relativistic simulations in this paper: $\ell_{1,2}/M_0 = 23.44, 25.94, 29.78, 35.72$. The corresponding orbital angular velocity parameters are $\Omega M_0 = 0.01547, 0.01296, 0.01022, 0.00746$.

In the following we first take care to construct invariant measures that can be used to compare, and give a sense of the difference between, two different spacetimes, one of which is obtained by solving the Einstein equations and the other represented by the CFQE-sequence, which does not satisfy the Einstein equations.

c. The QE approximation. One basic assumption of the CFQE approach is the existence of a timelike, helical Killing vector field. In the case of a corotating binary, the 4-velocity u^a of the fluid is proportional to this Killing vector. This implies the vanishing of the type-(0,2) 4-tensor $Q_{ab} \equiv \nabla_a u_b + \nabla_b u_a + u_a a_b + u_b a_a$, where $a^a \equiv u^b \nabla_b u^a$ is the 4-acceleration of the fluid (∇_a denotes the covariant derivative operator compatible with the 4-metric g_{ab}). The quantity $Q_{ab} u^a$ vanishes identically. We can thus monitor the space-space components of Q_{ab} during full general relativistic simulations as a way of monitoring how well the 4-velocity u^a stays proportional to a Killing vector field. Define Q_{ij} as the projection of Q_{ab} onto the constant t spatial slice: $Q_{ij} = P_i{}^a P_j{}^b Q_{ab} = Q_{ab} \left(\frac{\partial}{\partial x^i}\right)^a \left(\frac{\partial}{\partial x^j}\right)^b$, where $P_{ab} = g_{ab} + n_a n_b$ is the projection operator onto the constant t spatial slices with unit normal n^a. The 3-coordinate independent norm $|Q_{ij}|$ of this matrix Q_{ij} is the square root of the largest eigenvalue of $Q_{ij} Q^j{}_k$, where we have raised and lowered 3-indices with the 3-metric.

To gain a sense of the size of $|Q_{ij}|$, we normalize $|Q_{ij}|$ by the norms of its two principle parts: $Q \equiv |Q_{ij}| / \max\{ |Q_{1ij}|, |Q_{2ij}| \}$, where $Q_{1ab} \equiv \nabla_a u_b + \nabla_b u_a$, and $Q_{2ab} \equiv u_a a_b + u_b a_a$. A value of $Q = 0$ signifies that the 4-velocity of the fluid is proportional to a timelike Killing vector as in the CFQE-sequence, while a value of $Q = 1$ would signify that the assumption is maximally

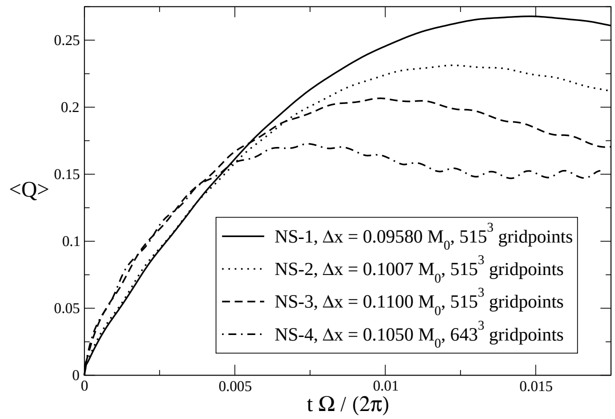

FIG. 1: The quantity $\langle Q \rangle$ is plotted as a function of time for fully consistent general relativistic numerical calculations, using CFQE configuration NS-1 to NS-4 as initial data.

violated.

Since Q is meaningful only inside the fluid bodies, a natural global measure of the magnitude of Q is its the baryonic mass weighted integral, denoted by $\langle Q \rangle$:

$$\langle Q \rangle = \frac{\int d^3x\ |Q|\, \sqrt{\gamma}\rho W}{\int d^3x\ \sqrt{\gamma}\rho W}, \tag{1}$$

where the integrals are taken to be over the entire spatial slice, with support only where $\rho \neq 0$. Here, we have denoted γ to be the determinant of the 3-metric and W to be the Lorentz factor of the fluid so that $\int d^3x\ \sqrt{\gamma}\rho W$ is the total (conserved) rest mass of the system.

In Fig. 1, we plot $\langle Q \rangle$ as a function of time in the general relativistic numerical simulations, using CFQE configurations NS-1 to NS-4 as initial data. For each initial CFQE configuration, the subsequent solution to the Einstein equations has $\langle Q \rangle$ reaching a maximum value after a very short time (between 0.5% and 1.5% of an orbit, $2\pi/\Omega$ being the orbital period corresponding to each CFQE configuration). These results are in contrast to the CFQE-sequence approximation, where $\langle Q \rangle$ is exactly zero. The rapid growth of $\langle Q \rangle$ to such a large value signals that, in the present case of corotating neutron stars, after just a short time the CFQE-sequence is very different from the spacetime described by the full Einstein equations even when the latter is started with the same initial data. Within a small fraction of an orbit, the 4-velocity of the fluid already is in no way proportional to a Killing vector.

Also, we see from Fig. 1 that the maximum value obtained by $\langle Q \rangle$ decreases from approximately $\langle Q \rangle = 0.27$ to $\langle Q \rangle = 0.17$ as the geodesic separation of the initial CFQE configuration increases from $\ell_{1/2}/M_0 = 23.44$ to $\ell_{1/2}/M_0 = 35.72$. In principle one can obtain an arbitrarily accurate CFQE configuration by starting the inspiral evolution with a large enough separation. While starting at a larger separation does *not* imply the CFQE-sequence will be more astrophysically relevant at late times, the fully relativistic simulation starting with it

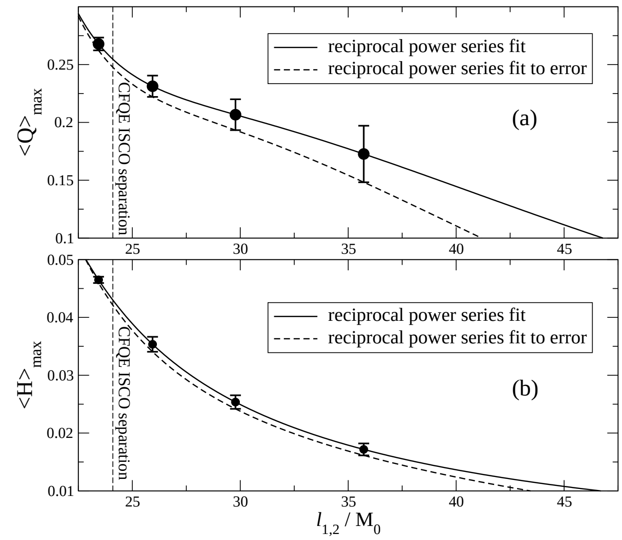

FIG. 2: Fig. 2a shows the maximum value of $\langle Q \rangle$ obtained in our fully consistent general relativistic simulations using CFQE configurations with different geodesic separation $\ell_{1,2}$ as initial data. Fig. 2b shows the corresponding maximum value of $\langle H \rangle$

will be. This effect can be seen directly in Fig. 2a, where the maximum value of $\langle Q \rangle$ obtained in the general relativistic simulations is plotted as a function of the initial geodesic separation $\ell_{1,2}$. A careful analysis of the numerical errors (shown as error bars in the figure) based on a Richardson extrapolation technique (for *both* truncation errors and boundary errors) can be found in [13]. In this way, we can determine the required separation of the initial CFQE configuration such that the QE assumption is valid to any required level in our fully consistent general relativistic simulations. In Fig. 2a, we fit a reciprocal power law, $\frac{a_1}{(\ell_{1,2})} + \frac{a_2}{(\ell_{1,2})^2} + \frac{a_3}{(\ell_{1,2})^3} + \frac{a_4}{(\ell_{1,2})^4}$, to the maximum value attained in our general relativistic calculations (as well as to the lower bound of the estimated error). We can see that, e.g., one would have to use a CFQE configuration initial data set with a geodesic separation between the neutron stars of about $\ell_{1,2} = 47\, M_0$ in order for the subsequent solution to have the fluid 4-velocity following a Killing vector to the 10% level.

d. The CF Approximation. While the 3-metric can be chosen to be conformally flat on any one time slice, the assumption that the 3-metric remain conformally flat for all time violates the dynamical Einstein equations. Again we begin with an invariant construction of a measure of the violation. Because conformal flatness is a property of a constant time slice, and because we are using the same slicing conditions in our simulation as that used in the CFQE-sequence approximation, we seek a 3-invariant (with respect to spatial coordinate transformations) to monitor the CF assumption. The Bach 3-tensor $B_{ijk} = 2\mathcal{D}_{[i}\left({}^{(3)}R_{j]k} - \frac{1}{4}\gamma_{j]k}{}^{(3)}R\right)$ can be shown to vanish if and only if the 3-metric γ_{ij} is conformally flat. The Cotton-York tensor, H_{ij}, is related to the 3-Bach tensor by $H_{ij} =$

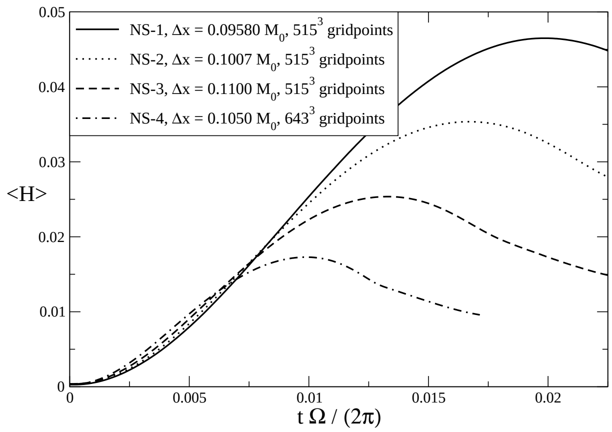

FIG. 3: The quantity $\langle H\rangle$ is plotted as a function of time for fully consistent general relativistic numerical calculations, using CFQE configuration NS-1 to NS-4 as initial data.

$\epsilon^{mn}{}_j B_{mni}$, where ϵ_{ijk} is the natural volume element 3-form. We define the scalar H as the matrix norm of H_{ij}, normalized by the size of the covariant derivative of the 3-Ricci tensor: $H = |H_{mn}|/\sqrt{\mathcal{D}_i{}^{(3)}R_{jk}\mathcal{D}^{i\,(3)}R^{jk}}$, where $|H_{ij}|$ denotes the matrix norm of the H_{ij}. As before, we define the baryonic mass density weight of H, denoted as $\langle H\rangle$, by

$$\langle H\rangle = \frac{\int d^3x\ |H|\,\sqrt{\gamma}\rho W}{\int d^3x\,\sqrt{\gamma}\rho W}, \tag{2}$$

where the integrals are taken to be over the entire spatial slice, but have support only where $\rho \neq 0$.

In Fig. 3, we plot $\langle H\rangle$ as a function of time for the general relativistic simulations using initial data CFQE configurations NS-1 through NS-4. The profile of the violation of the conformal flatness assumption $\langle H\rangle$ is similar to that of the Killing vector assumption $\langle Q\rangle$: a quick rise to a maximum value at a timescale on the same order of but slightly longer than that of $\langle Q\rangle$. This is contrasted to the CFQE-sequence approximation, which has $\langle H\rangle$ exactly equal to 0.

In Fig. 2b, we plot the maximum value of the quantity $\langle H\rangle$ as a function of initial geodesic separation $\ell_{1,2}$ attained in our general relativistic numerical simulations using the four CFQE configurations NS-1 through NS-4 as initial data (see [13] for a detailed analysis of the truncation and boundary errors in our calculation, which we indicate as error bars in the figure). As in Fig. 2a, we fit a reciprocal power law to both the maximum value of $\langle H\rangle$ and the lower bound of the error. We see that a CFQE configuration with geodesic separation $\ell_{1,2}$ of approximately 47 M_0 or greater must be used as initial data in order for the full solution to the Einstein field equations to have a value of $\langle H\rangle$ to be 0.01 or less.

e. Conclusions In this letter we present a method for analyzing the regime of validity of the CFQE-sequence approximation. We apply this method to the CFQE-sequence approximation of corotating binary neutron stars. We show that for geodesic separations less than roughly 45 M_0, which is about twice the geodesic separation of the ISCO configuration, the QE approximation is severely violated and the CFQE sequence cannot be taken as a reasonable approximation of the binary evolution for these separations. Numerical simulations starting with CFQE configurations as initial data with geodesic separations smaller than 45 M_0 cannot, therefore, be considered as approximating a realistic neutron star binary inspiral.

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